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Note

A Binomal Identity Related to Ballots and Trees*

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SECTION 1

In a recent paper [3], Riordan has obtained the binomial identity

$$\binom{rn+r}{n} = \sum \binom{n+1}{k} \frac{k!}{k_1! \cdots k_n!} \binom{r}{1}^{k_1} \cdots \binom{r}{n}^{k_n}, \tag{1}$$

where $k = k_1 + \cdots + k_n$ and the summation is over all nonnegative $k_1, \ldots, k_n$ such that

$$k_1 + 2k_2 + \cdots + nk_n = n. \tag{2}$$

(There is an unfortunate typographical error in the statement of this identity in [3].)

The object of the present note is to give a simple proof of (1) and indeed of the slightly more general fomula

$$\binom{nr+pr}{n} = \sum \binom{n+p}{k} \frac{k!}{k_1! \cdots k_n!} \binom{r}{1}^{k_1} \cdots \binom{r}{n}^{k_n}, \tag{3}$$

where $k = k_1 + \cdots + k_n$, the summation is over all nonnegative $k_1, \ldots, k_n$ satisfying (2) and p is arbitrary.

SECTION 2

We shall make use of the following formula proved in [1]:

$$\sum_{s_j=0}^{\infty} \frac{(p+2s_1+3s_2+4s_3+\cdots)!}{s_1!\, s_2!\, s_3! \cdots (p+s_1+2s_2+\cdots)!} \frac{u_1^{s_1} u_2^{s_2} \cdots}{(1+u_1+u_2+u_3+\cdots)^{2s_1+3s_2+\cdots}}$$

$$= \frac{(1+u_1+u_2+u_3+\cdots)^{p+1}}{1-u_1-2u_2-3u_3-\cdots} \qquad (p = 0, 1, 2, \ldots), \tag{4}$$

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where the series

$$u_1 + u_2 + u_3 + \cdots$$

is absolutely convergent. It follows easily from (4) that

$$\sum_{s_0, s_1, s_2, \ldots = 0}^{\infty} \frac{(p + s_0 + 2s_1 + 3s_2 + \cdots)!}{s_0!\, s_1!\, s_2! \cdots (p + s_1 + 2s_2 + \cdots)!}$$

$$\times \frac{u_0^{s_0} u_1^{s_1} u_2^{s_2} \cdots}{(1 + u_0 + u_1 + u_2 + \cdots)^{s_0 + 2s_1 + 3s_2 + \cdots}}$$

$$= \frac{(1 + u_0 + u_1 + u_2 + \cdots)^{p+1}}{1 - u_1 - 2u_2 - 3u_3 - \cdots} \qquad (p = 0, 1, 2, \ldots).$$

Moreover, if we take all $u_i = 0$ except u_{r-1}, (5) yields

$$\sum_{n=0}^{\infty} \binom{rn + p}{n} \frac{u^n}{(1 + u)^{rn}} = \frac{(1 + u)^{p+1}}{1 - (r - 1)u}, \tag{6}$$

a formula usually proved by means of the Lagrange-Bürmann expansion [2, p. 126, no. 216].

In (5) take

$$u_i = \binom{r}{i + 1} u^{i+1} \qquad (i = 0, 1, 2, \ldots). \tag{7}$$

Then clearly

$$1 + u_0 + u_1 + u_2 + \cdots = (1 + u)^r$$

and

$$1 - \sum_{i=1}^{\infty} i u_i = (1 + u)^{r+1}(1 - (r - 1)u).$$

Thus (5) becomes

$$\sum_{s_0, s_1, s_2, \ldots = 0}^{\infty} \frac{(p + s_0 + 2s_1 + 3s_2 + \cdots)!}{s_0!\, s_1!\, s_2! \cdots (p + s_1 + 2s_2 + \cdots)!} \frac{\binom{r}{1}^{s_0} \binom{r}{2}^{s_1} \cdots u^{s_0 + 2s_1 + 3s_2 + \cdots}}{(1 + u)^{r(s_0 + 2s_1 + 3s_2 + \cdots}}$$

$$= \frac{(1 + u)^{pr+1}}{1 - (r - 1)u}. \tag{8}$$

Now put

$$S(n, r, p) = \sum \binom{n + p}{k} \frac{k!}{k_1! \cdots k_n!} \binom{r}{1}^{k_1} \cdots \binom{r}{n}^{k_n}, \tag{9}$$

where $k = k_1 + \cdots + k_n$ and the summation is over all nonnegative $k_1, \ldots, k_n$ such that

$$k_1 + 2k_2 + \cdots + nk_n = n.$$

Then (8) implies

$$\sum_{n=0}^{\infty} S(n, r, p) \frac{u_n}{(1+u)^{rn}} = \frac{(1+u)^{pr+1}}{1-(r-1)u}. \tag{10}$$

By (10) and (6) we have

$$\sum_{n=0}^{\infty} S(n, r, p) \frac{u^n}{(1+u)^{rn}} = \sum_{n=0}^{\infty} \binom{nr+pr}{n} \frac{u^n}{(1+u)^{rn}}$$

and therefore

$$S(n, r, p) = \binom{nr+pr}{n}.$$

This completes the proof of (3).

References

1. L. Carlitz, Some summation formulas, *Fibonacci Quart.* **9** (1971), 28–34.
2. G. Polya and G. Szego, "Aufgaben und Lehrsatze aus der Analysis," I, Berlin, 1925.
3. John Riordan, Ballots and plane trees, *Journal of Combinatorial Theory* **11** (1971), 85–88.